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# *LR-Upward* Drawing: a More Usable Ordered Sets Drawing

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## Abstract

*In this paper, we introduce a new concept to draw an ordered set: the LR-Upward drawing is an upward drawing based on a chain decomposition of the order such that elements drawn on the same vertical line are always comparable and all other comparabilities flow from left to right. We describe a particular technique for automatically generating enhanced LR-Upward drawing for N-Free orders that are X-Cycle-Free. This technique first enhances locally the drawing, around a particular chain, and then expands the enhancement on the remaining part of the order. This technique can be used to enhance the usability of some complex pictures.*

## 1. Introduction

When a complex and potentially confusing information should be presented to a user, one option is to use a graph showing all the relevant elements and their relationships in a visual way. In some cases, the relationships between elements is a hierarchy (i.e. an antisymmetric and transitive relation), in which case the graph is an ordered set. Several methods for the automatic, computer generated drawing of ordered sets are available in the literature (see [2, 9, 11] for surveys on the question), but none of them is fully satisfying. For example, orders with large number of covering relations (i.e. with lots of edges to represent) are particularly difficult to render. The most common drawing technique used to represent ordered sets is the upward drawing (or Hasse diagram). An upward drawing suppresses all nonessential edges (those implied by transitivity and the loops at each vertex due to reflexivity) and draws only the directed covering graph of the order in such a way that covering relations are all directed upward. In other words, an upward drawing contains no horizontal edges, and no nonessential edges, and if an element $x$ is smaller than $y$ then there exists a path from $x$ to $y$ that is directed upwards.

Short of having a general definition of what a "good" upward drawing of a given order is, several standard criteria have been identified and analyzed in isolation or combined. These criteria include planarity [6, 5], the slope of the edges, the number of directions, the number of edge crossing [7] etc. Among the qualities that are expected from a good drawing, one obviously important one is how easy it is to see if two elements are comparable in the order.

In this paper we propose a modified version of the upward drawing: the *LR-Upward* drawing concept. It is a new way to visualize ordered sets. Our approach is to use a chain decomposition of the ordered set as the underlying structure for positioning the vertices of the order. We focus particularly on *N-Free* and *X-Cycle-Free* orders. For that class, we detail a complete technique to generate and improve automatically an *LR-Upward* drawing. A Java implementation of the algorithms described in this paper is available.

## 2. Definitions

An ordered set $P = (X, \leq)$ is a pair consisting of a nonempty set $X$ and a binary relation $\leq$ on $X$, satisfying *reflexivity*, *antisymmetry* and *transitivity*. Two elements $x$ and $y$ are *comparable* in $P$ if either $x \leq y$ ($x$ is a *predecessor* of $y$) or $y \leq x$ ($x$ is a *successor* of $y$); otherwise, $x$ and $y$ are *incomparable*, which we note $x \| y$. The covering relation of an ordered set $P$ is the transitive reflexive reduction of $P$. An element $y$ of $P$ *covers* another element $x$, and we note $x \prec y$ ($x$ is a *immediate predecessor* of $y$), if $\forall z, x \leq z \leq y$ and $x \neq z$ implies $z = y$.

A *chain* in $P$ is a set of pairwise comparable elements. An *antichain* in $P$ is a set of pairwise incomparable elements. The *width* of $P$, $width(P)$, is the size of its longest antichain. A *chain decomposition* of $P$ is a partition of $P$ into chains. Every order $P$ can be partitioned into $width(P)$ chains, and this is the smallest possible partition [3]. A *linear extension* of $P$ is a total ordering $x_1, x_2, \cdots, x_n$ of $P$ such that if $x_i \leq x_j$ in $P$ then $i \leq j$. Every order has linear extensions [13]. A linear extension $L = \{x_1 < x_2 < \cdots < x_n\}$ of $P$ is *greedy* if it is constructed inductively such that the $i + 1^{th}$ element is chosen minimal in $P \setminus \{x_1, \cdots, x_i\}$ so that it is comparable to $x_i$ whenever possible. Intuitively, a greedy linear extension is built by always "climbing as high as you can" along the chains.

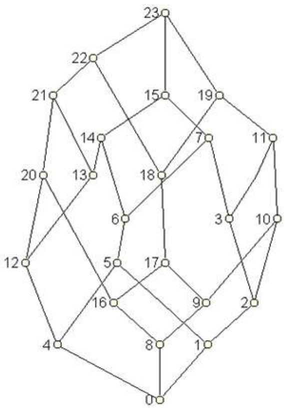

**Figure 1. A 24 vertice order drawing generated with LatDraw [4].**

## 3. *LR-Upward* drawing

The fundamental idea of *LR-Upward* drawing of an ordered set $P$ is to start from a chain decomposition of $P$ and draw each chain along a different vertical line. The goal is to have comparability information more implicit: if two elements are on the same vertical line, then they are comparable. This very simple idea provides a good visual rendering for some classes of ordered sets, *e.g.* the ones than can be decomposed into a small number of chains. However, this idea alone does not help when it comes to figuring out the comparability of two elements that are not on the same chain in the initial chain decomposition.

In order to improve the readability of the comparability of elements that are not on the same chain, we propose to use a chain decomposition derived from a linear extension of the order. Thanks to that choice, we are able to ensure that in addition to the usual "bottom to top" direction, all covering relations go from left to right (hence the name of *LR-Upward*, for "left to right, upward"). This technique introduces a sense of "flow" in the figure, which helps with the reading of the comparabilities. Figure 1 shows an example of an order drawn with the software LatDraw [4]. The same order drawn with our LR-Upward drawing software is shown Figure 2. As one can see, and in this case, the flow of information is much better rendered with the *LR-Upward* drawing.

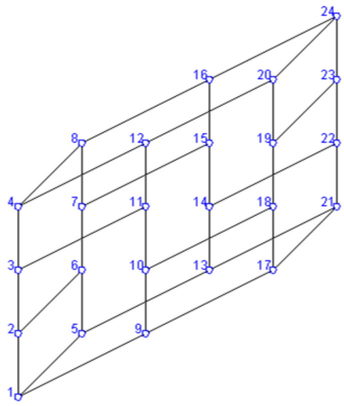

**Figure 2. An *LR-Upward* drawings of the order of Figure 1.**

The algorithm used to obtain an *LR-Upward* drawing of an ordered set $P$ can be informally sketched as follows ($C_i[k]$ stands for the $k^{th}$ element of the chain $C_i$, $|C_i|$ stands for the number of elements in the chain $C_i$, and $x_x$ and $x_y$ are the $x$ and $y$ coordinate of the element $x$ in the figure):

1: Generate an "appropriate" linear extension $L$ of $P$ that produce a decomposition of $P$ into a set of chains ($L = C_1 \oplus C_2 \oplus C_3 \cdots \oplus C_n$).
2: **for** $i = 1$ to $n$ **do**
3:   **for** $j = 1$ to $|C_i|$ **do**
4:     $x = C_i[j]$
5:     $x_x = i$
6:     **if** $x$ is minimal in $P$ **then**
7:       $x_y = 1$
8:     **else**
9:       $x_y = Max(\{y_y, y \prec x\}) + 1$
10:     **end if**
11:     Draw $x$ at location $(x_x, x_y)$
12:     Draw a line between $x$ and each of its immediate predecessors
13:   **end for**
14: **end for**

Because $L$ is a linear extension of $P$, it is clear that when $x$ is selected at line 4, all of its predecessors have been already selected, thus the line 12 can be achieved and the resulting edges go either vertically on chain $C_i$ or from left to right for the predecessors that are not in $C_i$.

## 4. *N-Free* and *X-Cycle-Free* ordered sets

We now investigate the *LR-Upward* drawing using the approach described above on *N-Free* and *X-Cycle-Free* ordered sets. *N-Free* orders define an important class in the theory of ordered sets that has been very well investigated due to its relations with many applications [10].

The $N$ ordered set is an ordered set of four distinct elements $a, b, c$ and $d$ such that $a \prec c$, $b \prec d$, $b \prec c$ and $a \| d$. An ordered set is *N-Free* if its diagram contains no sub-diagram isomorphic to $N$.

The $X$-cycle ordered set is an ordered set of four distinct elements $a, b, c$ and $d$ such that $a \prec c$, $b \prec d$, $b \prec c$ and $a \prec d$. An ordered set is *X-Cycle-Free* if its diagram contains no sub-diagram isomorphic to the $X$-cycle.

Every finite ordered set can be embedded into an *N-Free* and/or *X-Cycle-Free* ordered set in a linear time for an extra $O(n)$ space in the worst-case (e.g. by adding an element on each covering relation). Thus, an upward drawing solution for this family of orders is also an upward drawing solution for general orders (although it is not necessarily a straight edge upward drawing).

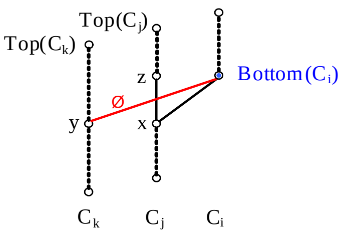


**Figure 3. Links from "Top" (a) and links to "Bottom" (b)**

### 4.1. Preliminary facts about *N-Free* and *X-Cycle-Free* ordered sets

We prove that *N-Free* and *X-Cycle-Free* ordered sets can be drawn using the *LR-Upward* Drawing in such a way that the only relationships between elements that are not on the same chain are between the bottom of a chain and a single smaller element, and between the top of the chain and a single larger element. The next two lemmas formally state this property.

**Lemma 1** *Let $P$ be a N-Free ordered set. Let $L$ be a greedy linear extension of $P$ where $L = C_1 \oplus C_2 \oplus C_3 \cdots \oplus C_n$. Let $C_i$ and $C_j$ be two chains from $L$ such that $i < j$. There are at most two possible links between the two chains:*

1. $(Top(C_i), v)$, *where* $v \in C_j \setminus Bottom(C_j)$ *(Fig. 3 (a))*
2. $(u, Bottom(C_j))$, *where* $u \in C_i \setminus Top(C_i)$ *(Fig. 3 (b))*

Lemma 1 shows that in an *LR-Upward* drawing based on a greedy linear extension of *N-Free* orders, any edge between two chains goes either to the *Top* of one chain or from the *Bottom* of one chain. In addition to that, we now show that if the order is also *X-Cycle-Free*, a given chain can only have one such a relation from its bottom element to another (smaller) chain, and one such relation from its top element to another (larger) chain. We formally define this property in this next lemma:

**Lemma 2** *Let $P$ be a finite N-Free ordered set which is also X-Cycle-Free. Let $L$ be a greedy linear extension of $P$ where $L = C_1 \oplus C_2 \oplus C_3 \cdots \oplus C_n$. $\forall i, 1 \leq i \leq n$, we have:*

1. *$Bottom(C_i)$ has at most one immediate predecessor (on a chain $C_j$, $j < i$)*
2. *$Top(C_i)$ has at most one immediate successor (on a chain $C_j$, $i < j$)*

### 4.2. The "chains interchange" technique

The chain interchanging is a systematic technique that allows us to manipulate the chains of a greedy linear extension without affecting its basic property of being greedy. It only works for *N-Free* ordered sets and does not affect the number of chains in the decomposition. This operation has been introduced by Rival [8] to successfully prove that any greedy linear extension is optimal for the jump number problem. Moreover, every optimal linear extension for the jump number problem is actually a greedy linear extension [12].

This technique is at the heart of our approach since it will allow us to "navigate" among the greedy linear extensions in order to find the "appropriate" one that could be used as the underlying structure for our upward drawing.

Let $P$ be an *N-Free* ordered set and let $L$ be a greedy linear extension of $P$ where $L = C_1 \oplus C_2 \oplus C_3 \cdots \oplus C_n$. For every index $i$ such that $0 \leq i < n$, there are at most two covering relations between the chains $C_i$ and $C_{i+1}$, that is, $Top(C_i) \prec u$ for some $u \in C_{i+1} \setminus Bottom(C_i{+}1)$ and $v \prec Bottom(C_{i+1})$ for some $v \in C_i \setminus Top(C_i)$.

Let $C'_i = \{x \in C_i : x \leq v\} \cup \{x \in C_{i+1} : x < u\}$ and $C'_{i+1} = \{x \in C_i : x > v\} \cup \{x \in C_{i+1} : x \geq u\}$.

We transform the greedy linear extension $L = C_1 \oplus C_2 \oplus \cdots \oplus C_i \oplus C_{i+1} \cdots \oplus C_n$ into another greedy linear extension $L = C_1 \oplus C_2 \oplus \cdots \oplus C'_i \oplus C'_{i+1} \cdots \oplus C_n$. We denote this operation by $IC(L, i) = L'$ (see Figure 4).

Notice that any of the four subsets in both definitions of $C'_i$ and $C'_{i+1}$ could be empty depending on the configuration between the chains $C_i$ and $C_{i+1}$.

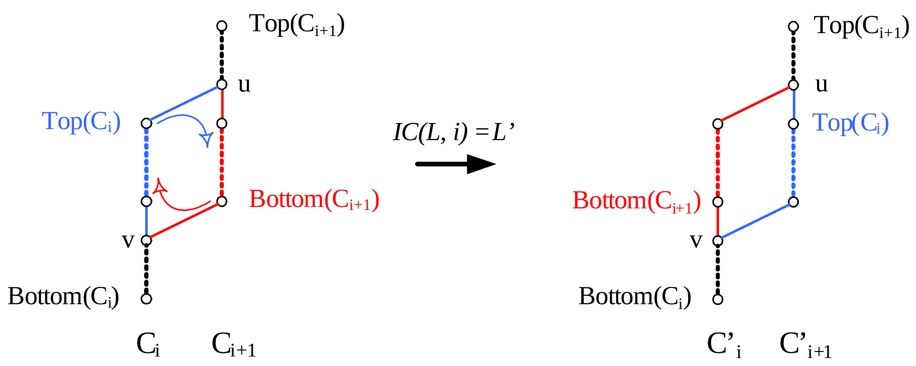


**Figure 4. The chains interchange between two consecutive chains.**

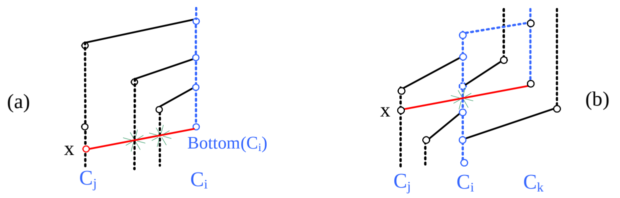


**Figure 5. Unresolved edge crossing after restructuring $C_i$.**

## 5. *LR-Upward* drawing of *N-Free* and *X-Cycle-Free* ordered sets

### 5.1. Local drawing improvement

Contrary to many conventional approaches, our aim is not to directly improve globally the upward drawing. Instead, we first approach the problem with a local improvement of certain parts of the drawing and then we recursively try to expand this local enhancement to the remaining parts of the diagram.

Our goal is to enhance the local display around a chain $C_i$: reduce the number of edges crossing and give a sense of a flow from the bottom left to the top right. To achieve this, we rearrange the chains on both sides of the chain $C_i$ so that these chains are ordered according to the position in which they are connected to $C_i$. In other words, we want to reach a chain decomposition $C_1 \oplus C_2 \oplus \cdots \oplus C_i \oplus \cdots \oplus C_n$ such that $\forall j, k \leq n$, if $\exists x, y \in C_i$ such that $x < y$, *Top*$(C_j) \prec x$ and *Top*$(C_k) \prec y$, then $j > k$. Conversely, if $\exists x, y \in C_i$ such that $x < y$, $x \prec$ *Bottom*$(C_j)$ and $y \prec$ *Bottom*$(C_k)$, then $j > k$.

The left-neighbors chains have to be ordered as displayed in the Figure 8. To accomplish this, we "pull" successively each left neighbor until it reaches its "targeted" place. We start the process by pulling the neighbor chain which is linked by the smaller element in $C_i$. The general formula for the position is as follows: if $C_k$ is linked by the $d^{th}$ element of $C_i$ (among these elements that are linked to a chain to the left), then the targeted place of $C_k$ is the position $i-d$. However, the place exchange between $C_k$ and the chain currently located at position $i-d$ cannot always be done immediately, since our chain exchange algorithm requires the chains to be consecutive in the linear extension. Consequently, we will need $i-d-k$ chain exchanges to achieve our goal.

Formally, let $L = C_1 \oplus C_2 \oplus \cdots \oplus C_i \oplus \cdots \oplus C_n$ be a greedy linear extension of $P$, and let $i < n$. We define the set *Left-Links*$(L, C_i)$ as the set of elements $x \in P$, such that $x =$ *Top*$(C_k)$ for some $k < i$ and $x$ is covered by some element in $C_i$. The reorganization of the left-neighborhood of a chain $C_i$ in a greedy linear extension can be established as follows:

**Lemma 3** *Let $P$ be a finite ordered set which is N-Free and X-Cycle-Free. Let $L = C_1 \oplus C_2 \oplus \cdots \oplus C_{i-1} \oplus C_i \oplus \cdots \oplus C_n$ be a greedy linear extension of $P$, and let $i \leq n$. There exists another greedy linear extension $L' = C'_1 \oplus C'_2 \oplus \cdots \oplus C'_{i-1} \oplus C_i \oplus \cdots \oplus C_n$ of P such that:*

1. *Left-Links$(L, C_i) =$ Left-Links$(L', C_i)$*
2. *If Top$(C'_k) < u$ and Top$(C'_j) < v$ for some elements $u, v \in C_i$ and $u < v$, then $j < k$*
3. *All chains having their top element in Left-Links$(L', C_i)$ are consecutive: let $j$ such that Top$(C'_j) \in$ Left-Links$(L', Ci)$, $\forall k$ such that $j \leq k < i$, Top$(C'_k) \in$ Left-Links$(L', C_i)$.*

Clearly, the same idea can be applied to the "right" side of the chain $C_i$.

#### 5.1.1 Limitations of the local restructuring

The local improvement technique presented above fails to resolve all the possible edge crossings around the chain

$C_i$ that is being reorganized. There are two situations in which some crossing will remain. In the first situation (Figure 5 (a)), the bottom (*resp.* the top) of $C_i$ is linked to another chain $C_j$ whose top is part of *Left-Links*$(L, C_i)$ (*resp.* *Right-Links*$(L, C_i)$). If $C_j$ is not immediately next to $C_i$, then a crossing may occur. The second situation occurs when there is a covering relation between a chain $C_j$ whose top is in *Left-Links*$(L, C_i)$ and a chain $C_k$ that is also covering the top of $C_i$ (Figure 5 (b)). The same situation may occur on the right of $C_i$.

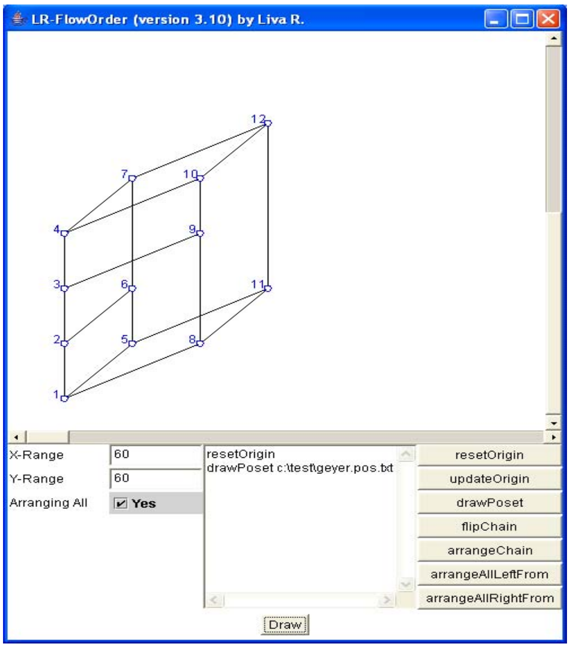


**Figure 6. A 12 vertice order as rendered by our tool.**

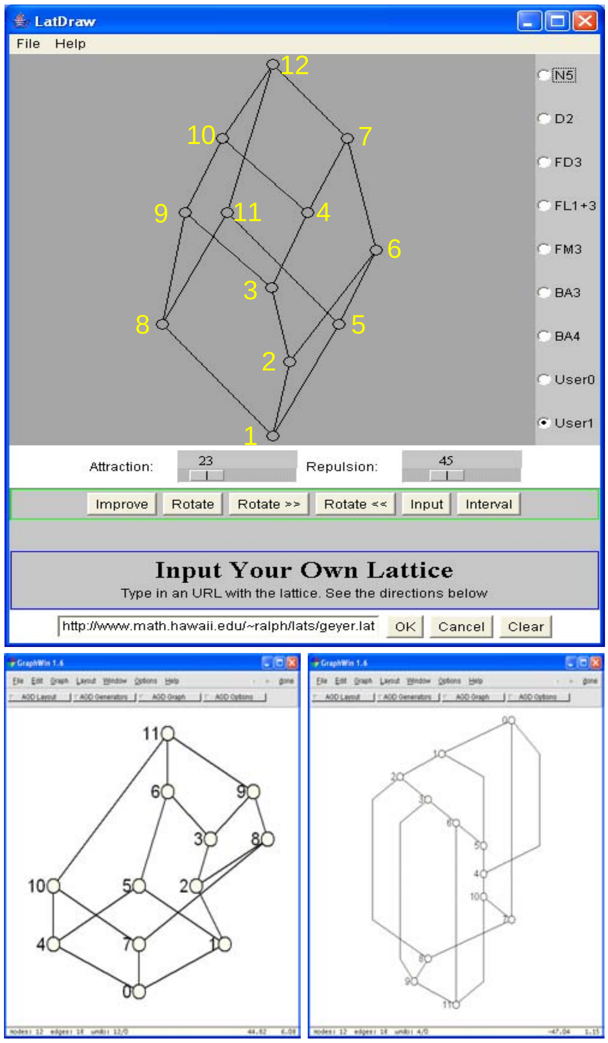


**Figure 7. The order of Figure 6 rendered with LatDraw [4] (top) and with GraphWin [1] (bottom).**

## 5.2. The global picture enhancement algorithm

One of the main issues with generalizing the local improvement presented above is that "improving" one chain may deteriorate the improvement already done on another chain. However, restructuring the "left" neighbors of a chain $C_j$ will never conflict with the left-neighborhood of a chain $C_i$ already arranged as long as $C_j$ is on the left of $C_i$ (i.e. $j < i$). Conversely, restructuring the "right" neighbors of a chain $C_k$ will never conflict with the right-neighborhood of a chain $C_i$ already arranged as long as $C_k$ is on the right of $C_i$ (i.e. $i < k$).

### 5.2.1 The "heaviest" chain approach

The technique we suggest to improve globally the drawing is to automatically select a chain that "splits" our drawing in two parts. Such a chain must have the most "connections" in terms of left-neighbors and right-neighbors. We choose that particular chain because it is the one that will most benefit from a "complete" local enhancement, and that will have the most chances of producing edge crossing.

Once such a starting chain $C_i$ is selected, we then restructure left-neighborhood "decreasingly" from the index $i$ to the first chain and the right-neighborhood "increasingly" from the index $i$ to the last chain. The index k of the heaviest chain can be computed beforehand during the initial drawing of $P$.

## 6. Conclusion

In this paper, we introduce a new technique for drawing ordered sets: the *LR-Upward* drawing is an upward drawing that has its edges flowing from the bottom left to the top right. The technique can be used on any ordered set, but we give a specific algorithm that is effective at producing an improved *LR-Upward* drawing for the ordered sets that are *N-Free* and *X-Cycle-Free*. A full Java implementation of the techniques introduced in this paper and several examples are available from `http://www.site.uottawa.ca/~zaguia`. The example of Figure 6 was produced with that software, while the same order is shown in Figure 7 rendered with two other tools, LatDraw [4] and GraphWin [1].

In future work, we intend to investigate in further details the properties of such a diagram. In particular, one intriguing direction is the possibility to omit in the figure all the vertical edges, since comparabilities on vertical lines are implied. Under that new context, what are the orders that are planar? Can it be decided in polynomial time?

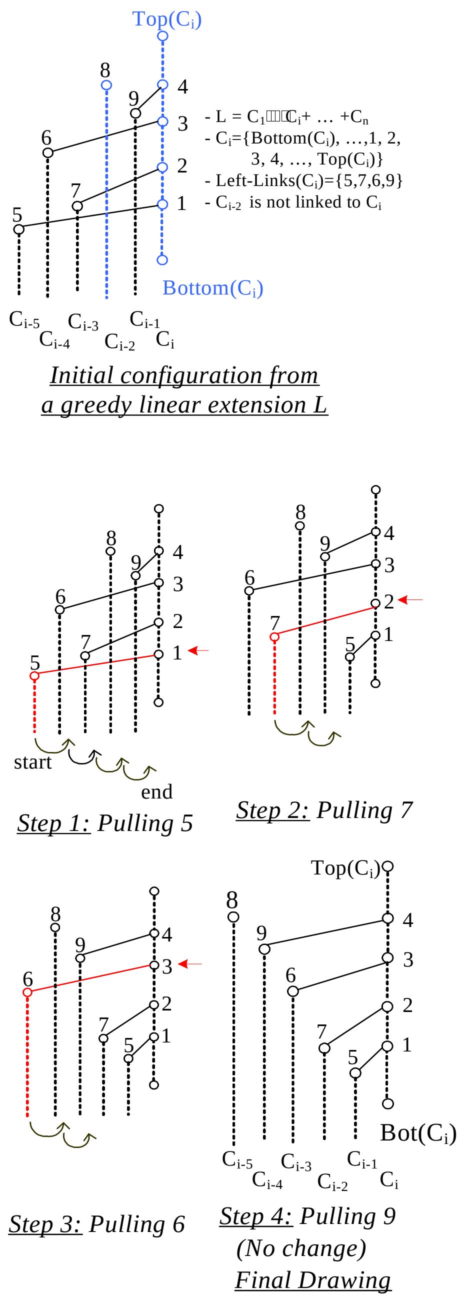


**Figure 8. The Left neighbors chains restructuring.**

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